

Calabi-Yau domains in three manifolds

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Abstract

We prove that for every smooth compact Riemannian three-manifold \overline{W} with nonempty boundary, there exists a smooth properly embedded one-manifold $\Delta \subset W = \text{Int}(\overline{W})$, each of whose components is a simple closed curve and such that the domain $\mathcal{D} = W - \Delta$ does not admit any properly immersed open surfaces with at least one annular end, bounded mean curvature, compact boundary (possibly empty) and a complete induced Riemannian metric.

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1 Introduction.

A natural question in the global theory of minimal surfaces, first raised by Calabi in 1965 [2] and later revisited by Yau [11, 12], asks whether or not there exists a complete immersed minimal surface in a bounded domain \mathcal{D} in \mathbb{R}^3 . As is customary, we will refer to this problem as the Calabi-Yau problem for minimal surfaces. In 1996, Nadirashvili [10] provided the first example of a complete, bounded, immersed minimal surface in \mathbb{R}^3 . However, Nadirashvili's techniques did not provide properness of such a complete minimal immersion in any bounded domain. Under certain restrictions on \mathcal{D} and the topology of an open surface¹ M , Alarcón, Ferrer, Martín, and Morales [1, 7, 8, 9] proved the existence of a complete, proper minimal immersion of M in \mathcal{D} . Recently, Ferrer, Martín and Meeks [4] have given a complete solution to the “**proper Calabi-Yau problem for smooth bounded domains**” by demonstrating that for every smooth bounded domain $\mathcal{D} \subset \mathbb{R}^3$ and for every open surface M , there exists a complete proper minimal immersion $f: M \rightarrow \mathcal{D}$; furthermore, in [4], they proved that such an immersion $f: M \rightarrow \mathcal{D}$ can be constructed so that for any two distinct ends E_1, E_2 of M , the limit sets $L(E_1), L(E_2)$ in $\partial\mathcal{D}$ are disjoint compact sets².

In contrast to the above existence results, in this paper we prove the existence of nonsmooth bounded domains \mathcal{D} in \mathbb{R}^3 , and more generally, domains \mathcal{D} inside any Riemannian three-manifold, for which some open surface M can not be properly immersed into \mathcal{D} as a complete surface with bounded mean curvature. In this case, we will say that \mathcal{D} is a **Calabi-Yau domain** for M . The result described

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¹We say that a surface is *open* if it is connected, noncompact and without boundary.

²See Definition 2.1 for the definition of the limit set of an end of a surface in a three-manifold.

in the next theorem generalizes the main theorem of Martín, Meeks and Nadirashvili in [6] which demonstrates the existence of nonsmooth bounded domains in \mathbb{R}^3 which do not admit any complete, properly immersed minimal surfaces with compact boundary (possibly empty) and at least one annular end.

Theorem 1.1 *Let \overline{W} be a smooth compact Riemannian three-manifold with nonempty boundary and let $W = \text{Int}(\overline{W})$. There exists a properly embedded one-manifold $\Delta \subset W$ whose path components are smooth simple closed curves, such that $\mathcal{D} = W - \Delta$ is a Calabi-Yau domain for any surface with compact boundary (possibly empty) and at least one annular end. In particular, \mathcal{D} does not admit any complete, noncompact, properly immersed surfaces of finite topology, compact boundary and constant mean curvature.*

2 Notation and the description of Δ .

Before proceeding with the proof of the main theorem, we fix some notation.

1. $\mathbb{B}(R) = \{x \in \mathbb{R}^3 \mid |x| < R\}$ and $\mathbb{B} = \mathbb{B}(1)$.
2. $\overline{\mathbb{B}(R)} = \{x \in \mathbb{R}^3 \mid |x| \leq R\}$ and $\overline{\mathbb{B}} = \overline{\mathbb{B}(1)}$.
3. $\mathbb{S}^2(R) = \partial\mathbb{B}(R)$ and $\mathbb{S}^2 = \partial\mathbb{B}$.
4. For $p \in \mathbb{R}^3$ and $\varepsilon > 0$, $\mathbb{B}(p, \varepsilon) = \{x \in \mathbb{R}^3 \mid d(p, x) < \varepsilon\}$ is the open ball of radius ε centered at p .
5. For $n \in \mathbb{N}$, $\mathbb{B}_n = \mathbb{B}(1 - \frac{1}{2^n})$ and $\mathbb{S}_n^2 = \partial\mathbb{B}_n$.
6. For any set $F \subset \mathbb{R}^3$, the cone on F is

$$C(F) = \{x \in \mathbb{R}^3 \mid x = ta \text{ where } t \in (0, \infty) \text{ and } a \in F\}.$$

7. For any set $F \subset \mathbb{R}^3$ and $\varepsilon > 0$, let $F(\varepsilon) = \{x \in \mathbb{R}^3 \mid d(x, F) \leq \varepsilon\}$ be the closed ε -neighborhood of F , where d is the distance function in \mathbb{R}^3 .

In the proof of Theorem 1.1, we will need the following definition.

Definition 2.1 *Let $f: M \rightarrow \mathcal{D}$ be a proper immersion of surface M with possibly nonempty boundary into an open domain \mathcal{D} contained in a three-manifold N with possibly nonempty boundary. The **limit set** of M is*

$$L(M) = \bigcap_{\alpha \in I} \overline{(f(M) - f(E_\alpha))},$$

where $\{E_\alpha\}_{\alpha \in I}$ is the collection of compact subdomains of M and the closure $\overline{f(M) - f(E_\alpha)}$ is taken in N . The **limit set** $L(e)$ of an end e of M is defined to be the intersection of the limit sets all properly embedded subdomains of M with compact boundary which represent e . Notice that $L(M)$ and $L(e)$ are closed sets of $\partial\mathcal{D}$, and so each of these limit sets is compact when N is compact.

First we will prove Theorem 1.1 in the case \overline{W} is the smooth closed Riemannian ball $\overline{\mathbb{B}} \subset \mathbb{R}^3$. In this case, we will construct a properly embedded 1-manifold $\Delta \subset \mathbb{B}$ with path components consisting of smooth simple closed curves such that every proper immersion $f: A = \mathbb{S}^1 \times [0, \infty) \rightarrow \mathbb{B} - \Delta$ of an annulus with a complete induced metric has unbounded mean curvature; this result will prove Theorem 1.1 in the special case $\overline{W} = \overline{\mathbb{B}}$. The proof of the case of Theorem 1.1 when \overline{W} is a smooth Riemannian ball, or more generally, an arbitrary compact smooth Riemannian manifold with nonempty boundary follows from straightforward modifications of the proof of the $\mathbb{B} - \Delta$ case; these modifications are outlined in the last paragraph of the proof.

The first step in the construction of Δ is to create a CW-complex structure Λ on the open ball \mathbb{B} . Consider the boundary ∂ of the box $[-1, 1] \times [-1, 1] \times [-1, 1] \subset \mathbb{R}^3$. The surface ∂ has a natural structure of a simplicial complex \mathcal{X}_1 with faces $\mathcal{F}_1 = \{F_1, F_2, \dots, F_6\}$ contained in planes parallel to the coordinate planes, edges $\mathcal{E}_1 = \{E_1, E_2, \dots, E_{12}\}$ and vertices $\mathcal{V}_1 = \{v_1, v_2, \dots, v_8\}$. Let \mathcal{X}_2 denote the related refined simplicial complex obtained from \mathcal{X}_1 by adding vertices to the centers of each of the faces of \mathcal{F}_1 and to the centers of each of the edges in \mathcal{E}_1 , thereby obtaining new collections $\mathcal{F}_2, \mathcal{E}_2, \mathcal{V}_2$ of faces, edges, and vertices. In this subdivision each face of \mathcal{F}_2 corresponds to subsquare in one the faces in \mathcal{F}_1 with four line segments, each of length one. Note that \mathcal{F}_2 has $6 \cdot 4$ faces, \mathcal{E}_2 has $2 \cdot 6 \cdot 4$ edges and \mathcal{V}_2 has $6 \cdot 4 + 2$ vertices. Continuing inductively the refining of the complex \mathcal{X}_2 , produces at the n -th stage a simplicial complex \mathcal{X}_n with $6 \cdot 4^{n-1}$ square faces \mathcal{F}_n , $2 \cdot 6 \cdot 4^{n-1}$ edges \mathcal{E}_n and $6 \cdot 4^{n-1} + 2$ vertices \mathcal{V}_n .

We define the 1-skeleton Γ of Λ as follows:

$$\Gamma = \bigcup_{k=1}^{\infty} [C(\mathcal{E}_k) \cap \mathbb{S}_k^2] \cup [C(\mathcal{V}_k) \cap (\overline{\mathbb{B}}_{k+1} - \mathbb{B}_k)],$$

where $C(\mathcal{E}_k)$ denotes the cone $C(\cup \mathcal{E}_k)$. Extend the proper 1-dimensional CW-complex $\Gamma \subset \mathbb{B}$ to a proper 2-dimensional CW-subcomplex Λ' of Λ as follows. The faces of Λ' are the spherical squares in $\mathbb{S}_k^2 - \Gamma$, as k varies in \mathbb{N} , together with the set of flat rectangles $C(\alpha) \cap (\mathbb{B}_{k+1} - \mathbb{B}_k)$, where α is a 1-simplex in $\Gamma \cap \mathbb{S}_k^2$, as k varies in \mathbb{N} and α varies in $\Gamma \cap \mathbb{S}_k^2$, see Figure 1 below. Let \mathcal{F} denote the set of faces of Λ . Finally, $\mathbb{B} - \Lambda'$ contains an infinite collection $\mathcal{G} = \{G_\alpha\}_{\alpha \in I}$ of components which have the appearance of a cube which is a radial product of a spherical square in some $\mathbb{S}_k^2 - \Gamma$ with a small interval of length $2^{-(k+1)}$, together with the special component $\mathbb{B}(\frac{1}{2})$. The set \mathcal{G} is the set of 3-cells in Λ , which completes the construction of the CW-complex structure Λ of \mathbb{B} .

Define the related closed, piecewise smooth regular neighborhood $\widehat{N}(\Gamma)$ of Γ :

$$\widehat{N}(\Gamma) = \bigcup_{k=1}^{\infty} \left[(C(\mathcal{E}_k) \cap \mathbb{S}_k^2) \left(\frac{1}{2^k 10} \right) \right] \cup \left[(C(\mathcal{V}_k) \cap (\overline{\mathbb{B}}_{k+1} - \mathbb{B}_k)) \left(\frac{1}{2^k 100} \right) \right].$$

Then let $N(\Gamma) \subset \text{Int}(\widehat{N}(\Gamma))$ be a small smooth closed regular neighborhood of Γ in \mathbb{B} such that its boundary $\partial N(\Gamma)$ intersects each face F in \mathcal{F} transversely in a simple closed curve $\beta(F)$ that bounds a disk $L(F) \subset F$; let $\mathcal{L} = \{L(F) \mid F \in \mathcal{F}\}$. For each open 1-simplex $\alpha \in \Gamma$, let $P(\alpha)$ be the plane perpendicular to α at the midpoint of α . Let $\widetilde{N}(\Gamma) \subset \text{Int}(\widehat{N}(\Gamma))$ be another smooth closed regular neighborhood of Γ with $N(\Gamma) \subset \text{Int}(\widetilde{N}(\Gamma))$ and such that $\partial \widetilde{N}(\Gamma) \cap P(\alpha)$ contains a simple closed curve $\beta(\alpha)$ close to α and which links α . Let $W(\alpha) \subset P(\alpha)$ denote the closed disk with boundary curve $\beta(\alpha)$ and let $\mathcal{W} = \{W(\alpha) \mid \alpha \in \Gamma\}$, see Figure 2.

The set Δ is the collection $[\bigcup_{\alpha \in \Gamma} \beta(\alpha)] \cup [\bigcup_{F \in \mathcal{F}} \beta(F)]$. The domain described in Theorem 1.1 is $\mathcal{D} = \mathbb{B} - \Delta$.

We conclude this section with the following immediate consequence of our constructions above.

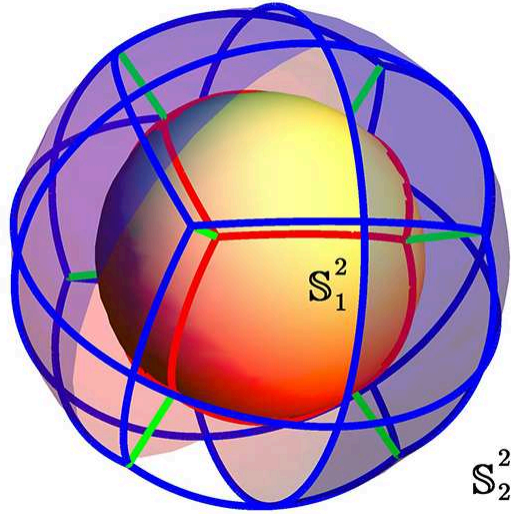


Figure 1: The first two steps in the construction of Λ .

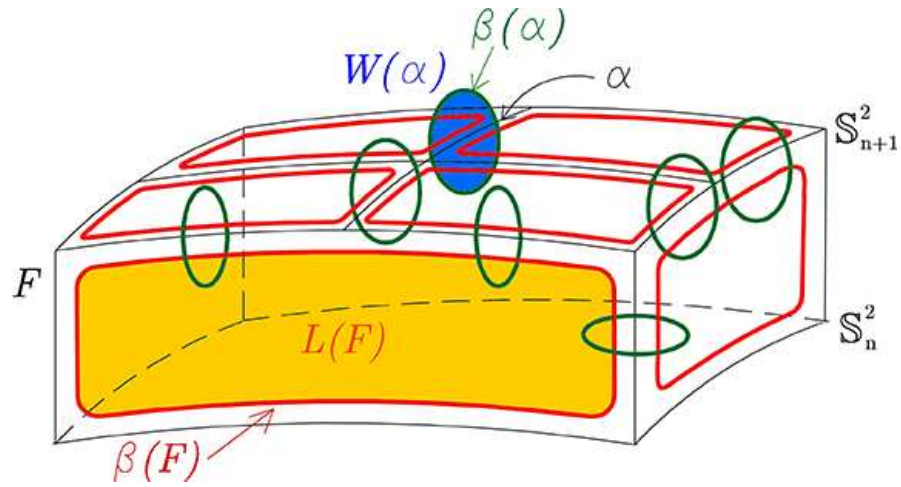


Figure 2: The 1-dimensional simplicial complex Γ , the 1-manifold Δ consisting of closed curves $\beta(F)$ and $\beta(\alpha)$ and the disks $W(\alpha)$ and $L(F)$, where F is a face in Λ and α is a 1-simplex in Λ .

Lemma 2.2 *Let E be one of the following:*

1. *a 1-simplex, face or 3-cell in Λ ;*
2. *a disk in either \mathcal{W} or \mathcal{L} ;*
3. *a component of $\tilde{N}(\Gamma) - \cup \mathcal{W}$.*

If for some $\delta \in (0, \frac{1}{4})$, $E \cap [\mathbb{B} - \mathbb{B}(1 - \delta)] \neq \emptyset$, then E is contained in a ambient ball B_E of radius 4δ .

3 $L(A)$ is a path connected subset of \mathbb{S}^2 with more than one point.

In this and the following sections, $f: A \rightarrow \mathcal{D}$ will denote a counterexample to Theorem 1.1 which, after a small smooth perturbation, we will assume to be a fixed properly immersed annulus diffeomorphic to $\mathbb{S}^1 \times [0, 1)$ satisfying:

1. The supremum of the absolute mean curvature of A is less than a fixed constant $H_0 > 10$;
2. f is transverse to the disks in \mathcal{W} and to the surface $\partial \tilde{N}(\Gamma)$;
3. f is in general position with respect to Λ , i.e., f is disjoint from the set of vertices \mathcal{V} of Λ , transverse to the closed faces of Λ and so, it is also transverse to \mathbb{S}_k^2 for each $k \in \mathbb{N}$.

Lemma 3.1 *If $f: \Sigma \rightarrow \mathcal{D}$ is a properly immersed surface with compact boundary and \mathbf{e} is an end of Σ , then the limit set $L(\mathbf{e})$ of the end \mathbf{e} is path connected.*

Proof. This is a standard result, but for the sake of completeness, we present its proof. Let $p, q \in L(\mathbf{e})$ be distinct points. Let $\mathcal{D}_1 \subset \mathcal{D}_2 \subset \dots \subset \mathcal{D}_n \subset \dots$ be a smooth compact exhaustion of \mathcal{D} . After replacing by subsequences, we may assume that there is a sequence of pairs of points p_n, q_n which lie in the component of $\Sigma - \text{Int}(f^{-1}(\mathcal{D}_n))$ which represents \mathbf{e} and such that $\lim_{n \rightarrow \infty} f(p_n) = p$ and $\lim_{n \rightarrow \infty} f(q_n) = q$.

Let $\sigma_n: [0, 1] \rightarrow \Sigma - \text{Int}(f^{-1}(\mathcal{D}_n))$ be paths with $\sigma_n(0) = p_n$ and $\sigma_n(1) = q_n$. Since the space $\mathcal{C}([0, 1], \overline{\mathbb{B}})$ of continuous maps of $[0, 1]$ into $\overline{\mathbb{B}}$ is a compact metric space in the sup norm, a subsequence of the paths $f \circ \sigma_n$ converges to a continuous map $f \circ \sigma$ of $[0, 1]$ to $\partial \mathcal{D} = [\mathbb{S}^2 \cup \Delta] \subset \overline{\mathbb{B}}$ with $f \circ \sigma(0) = p$ and $f \circ \sigma(1) = q$. Since $f \circ \sigma([0, 1]) \subset L(\mathbf{e})$ also holds, $L(\mathbf{e})$ is path connected. \square

Lemma 3.2 *If $L(A) \cap \Delta \neq \emptyset$ or if $L(A)$ consists of a single point in \mathbb{S}^2 , then A has finite area.*

Proof. By Theorems 3.1 and 3.1' in [3], the bounded mean curvature hypothesis and the properness hypothesis on f imply that if f composed with the inclusion map of \mathcal{D} into \mathbb{R}^3 is proper outside of a point in \mathbb{S}^2 or outside of a component of Δ , then the surface A has finite area. Since $L(A)$ is path connected and the path components of $\partial \mathcal{D}$ are \mathbb{S}^2 or a simple closed curve in Δ , then the lemma follows. \square

Lemma 3.3 *If $F: A \rightarrow \mathbb{R}^3$ is a complete immersion of $\mathbb{S}^1 \times [0, \infty)$ with bounded mean curvature, then A has infinite area.*

Proof. Suppose that A has finite area and we will obtain a contradiction. Since A is a complete annulus of finite area, there exists a sequence γ_n of pairwise disjoint, piecewise smooth, closed embedded geodesics with a single corner, which are topologically parallel to ∂A and whose lengths tend to 0 as n tends to infinity. Assume that the index ordering of the geodesics γ_n agrees with the relative distances of these curves to ∂A . Replace A by the subend $A(\gamma_1)$ with $\partial A(\gamma_1) = \gamma_1$. By the Gauss-Bonnet formula applied to the subannulus $A(\gamma_1, \gamma_n)$ with boundary $\gamma_1 \cup \gamma_n$, the total Gaussian curvature of $A(\gamma_1, \gamma_n)$ is greater than -4π . Since the Gaussian curvature function K_A of A is pointwise bounded from above by H_0^2 , then the integral $\int_{A(\gamma_1)} K_A^+ dA$, where $K_A^+(x) = \max\{K_A(x), 0\}$, is finite because A has finite area. Hence, after replacing A by a subend of A , we may assume that $\int_{A(\gamma_1)} K_A^+ dA < \pi$. So, we conclude that $\int_{A(\gamma_1, \gamma_n)} K_A^- dA > -5\pi$, for all n , where $K_A^-(x) = \min\{K_A(x), 0\}$.

On the other hand, since the area of A does not grow at least linearly with the distance from ∂A , the norm of the second fundamental form of A is unbounded on A . By standard rescaling arguments (see for example [5]), there exists a divergent sequence $p_n \in A(\gamma_1)$ of blow-up points on the scale of the second fundamental form with norm of the second fundamental form at p_n being $\lambda_n > n$, and intrinsic neighborhoods $B_A(p_n, \frac{\lambda_n}{10})$ such that a subsequence of the rescaled surfaces $\lambda_n [f(B_A(p_n, \frac{\lambda_n}{10})) - p_n]$ converges in the C^2 -norm to a minimal disk D in \mathbb{R}^3 satisfying:

1. The norm of the second fundamental form of D is at most 1 and equal to 1 at the origin.
2. D is a graph over the projection to its tangent plane at the origin.
3. The total curvature of D is $-\varepsilon$ for some $\varepsilon > 0$. Hence for n large, the integral of the function K_A^- on $B_A(p_n, \frac{\lambda_n}{10})$ is less than $-\frac{\varepsilon}{2}$.

By property 3 above, we conclude that $\lim_{n \rightarrow \infty} \int_{A(\gamma_1, \gamma_n)} K_A^- dA = -\infty$, which contradicts our earlier observation that $\int_{A(\gamma_1, \gamma_n)} K_A^- dA$ is bounded from below by -5π . \square

The next lemma is an immediate consequence of Lemmas 3.2 and 3.3.

Lemma 3.4 $L(A)$ is a path connected compact subset of \mathbb{S}^2 containing two distinct points x and y . In particular, the immersion f can be seen as a proper immersion in \mathbb{B} .

In the next sections, we will analyze how certain subdomains of the immersed annulus $f(A)$ intersects certain specific two-dimensional subsets of \mathcal{D} , for which we need the following definitions.

Definition 3.5 Suppose $F: \Sigma \rightarrow \mathcal{D}$ is a smooth proper immersion of a surface with compact boundary which is transverse to the disks in \mathcal{W} , to $\partial \tilde{N}(\Gamma)$ and is in general position with respect to Λ . Suppose γ is a simple closed curve in Σ . Then:

1. γ is an X_1 -type curve, if γ is a component of $F^{-1}(\cup \mathcal{W})$.
2. γ is an X_2 -type curve, if γ is a component of $F^{-1}(\partial \tilde{N}(\Gamma))$. Note that in this case $\gamma \subset [\partial \tilde{N}(\Gamma) - \cup \mathcal{W}]$ and so curves of X_1 -type and X_2 -type are disjoint.
3. $\gamma \subset \Sigma$ is an X_3 -type curve, if γ is a component of $F^{-1}(\cup \mathcal{L})$. Notice that in this case γ is contained in a face of Λ .

Definition 3.6 Given the fixed immersion $f: A \rightarrow \mathcal{D}$, then:

1. X_1 is the set of X_1 -type curves parallel to ∂A and X_2 is the set of X_2 -type curves parallel to ∂A .
2. X_3 is the set of X_3 -type curves in A which are disjoint from $(\cup X_1) \cup (\cup X_2)$.
3. By Lemma 4.1 below, the countable set X can be expressed as $X = X_1 \cup X_2 \cup X_3 = \{\gamma_i \mid i \in \mathbb{N}\}$, where the natural ordering of the simple closed, pairwise-disjoint curves γ_i in A by their relative distances from ∂A agrees with the ordering of the index set \mathbb{N} .
4. A_n denotes the compact subannulus in A with $\partial A_n = \partial A \cup \gamma_n$; note $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ is a smooth compact exhaustion of A .
5. For $n, j \in \mathbb{N}$, $A(n, j)$ denotes the compact subannulus of A with boundary curves γ_n and γ_{n+j} .
6. $A(k) = \cup_{j=1}^{\infty} A(k, j)$ is the end representative of A with boundary γ_k .

4 Placement properties of $\partial A(k, 1)$ for k large.

Lemma 4.1 *For k large, there exists at least one curve in X in the region $\mathbb{B}_{k+2} - \overline{\mathbb{B}_{k-1}}$. In particular, the set X is infinite.*

Proof. Assume that $f(\partial A)$ is contained in \mathbb{B}_n and we will prove that $\mathbb{B}_{k+2} - \overline{\mathbb{B}_{k-1}}$ contains an element in X , whenever $k > n$. Since $f: A \rightarrow \mathbb{B}$ is proper and transverse to the spheres \mathbb{S}_i^2 for every i , then for $i \geq n$, $f^{-1}(\mathbb{S}_i^2)$ contains a simple closed curve α_i which is parallel to ∂A . If $f(\alpha_k) \cap (\cup \mathcal{L}) \neq \emptyset$, then either $\alpha_k \in X_3$ or α_k intersects an element γ of $X_1 \cup X_2$, where $f(\gamma)$ is contained in $[\mathbb{B}_{k+1} - \overline{\mathbb{B}_{k-1}}] \subset [\mathbb{B}_{k+2} - \overline{\mathbb{B}_{k-1}}]$. Similarly, if $f(\alpha_{k+1}) \cap (\cup \mathcal{L}) \neq \emptyset$, then either $\alpha_{k+1} \in X_3$ or α_{k+1} intersects an element γ of $X_1 \cup X_2$, whose image $f(\gamma)$ must be contained in $[\mathbb{B}_{k+2} - \overline{\mathbb{B}_k}] \subset [\mathbb{B}_{k+2} - \overline{\mathbb{B}_{k-1}}]$. Hence, we may assume that $f(\alpha_k)$ and $f(\alpha_{k+1})$ are both disjoint from $\cup \mathcal{L}$ and so, $[f(\alpha_k \cup \alpha_{k+1})] \subset \text{Int}(\tilde{N}(\Gamma))$.

Let $D_{\mathcal{W}}^k$ be the collection of disks in \mathcal{W} which are contained in $\mathbb{B}_{k+1} - \overline{\mathbb{B}_k}$ and let Σ^k be the compact domain which is closure of the component of $\tilde{N}(\Gamma) - (\cup D_{\mathcal{W}}^k)$ which contains $f(\alpha_k)$ in its interior. Let $A(\alpha_k, \alpha_{k+1})$ be the subannulus of A with boundary $\alpha_k \cup \alpha_{k+1}$. Then $(f|_{A(\alpha_k, \alpha_{k+1})})^{-1}(\partial \Sigma^k)$ contains a simple closed curve γ which is parallel to ∂A and which is an element of $X_1 \cup X_2 \subset X$. The existence of γ completes the proof of the assertion. \square

Lemma 4.2 *There exists a small $\eta_1 > 0$ such that for any $\eta \in (0, \eta_1]$, if $D \subset A$ is a compact disk with $f(\partial D) \subset \mathbb{B}(z, \eta)$ for some $z \in \mathbb{S}^2$ and D contains a point p such that the distance $d(f(p), z) \geq 1$, then:*

1. *The disk D contains a X_i -type curve β , for $i = 1, 2$ or 3 , and $f(\beta)$ lies in $\mathbb{B}(z, 1/2) - \overline{\mathbb{B}(z, 2\eta)}$.*
2. *The curve β can be chosen so that the disk $D(\beta) \subset D$ bounded by β contains p . In particular, $f(D(\beta))$ contains a point of distance at least $\frac{1}{2}$ from its boundary and every point in $D(\beta)$ has intrinsic distance at least η from ∂D .*

Proof. Recall that for any face F in \mathcal{F} , $C(F)$ denotes the cone over F . Clearly, for $\eta_1 > 0$ sufficiently small and $\eta \in (0, \eta_1]$, there exist faces F_1, F_2, F_3 and F_4 in \mathcal{F} , such that: $\mathbb{B}(z, 2\eta) \subset \text{Int}(C(F_1))$, $C(F_i) \subset \text{Int}(C(F_{i+1}))$, for $i = 1, 2, 3$ and $C(F_4) \subset \mathbb{B}(z, 1/2)$.

At this point we can follow the proof of Lemma 4.1 where the annulus $D - \{p\}$ plays the role of A and the piecewise smooth disk $\partial C(F_i)$ plays the role of \mathbb{S}_{k-2+i}^2 . Then we obtain an X_i -type curve β parallel to ∂D in $D - \{p\}$ and whose image $f(\beta)$ is in the open region between $\partial(C(F_1))$ and $\partial(C(F_4))$, which is contained $\mathbb{B}(z, 1/2) - \overline{\mathbb{B}}(z, 2\eta)$. This is the desired curve. \square

Before stating the next assertion, we need some notation.

Definition 4.3 *Given a curve γ_k in X , we define $\chi_1(f(\gamma_k))$ to be the union of all closed 3-cells in Λ which intersect $f(\gamma_k)$. Similarly, given $i \in \mathbb{N}$ we define $\chi_{i+1}(f(\gamma_k))$ as the union of all closed 3-cells in Λ which intersect $\chi_i(f(\gamma_k))$.*

In what follows, we shall use the observation that for $i = 1$ and 2 , the set $\chi_i(f(\gamma_k))$ is a piecewise smooth compact ball, whose boundary sphere is a union of faces in \mathcal{F} and it is in general position with respect to the immersion f .

Lemma 4.4 *For k large, we have $f(\gamma_{k+1}) \subset \chi_3(f(\gamma_k))$ or $f(\gamma_k) \subset \chi_3(f(\gamma_{k+1}))$. Furthermore, given $\eta > 0$, there exists an integer $k(\eta)$ such that for any $k \geq k(\eta)$ one has:*

1. $f(A(k)) \subset [\mathbb{B} - \overline{\mathbb{B}}(1 - \eta)]$ and each X_i -type curve γ , $i = 1, 2$ or 3 , in $A(k, 1)$ is contained in a ball $\mathbb{B}(y(\gamma), \eta)$ for a suitable point $y(\gamma) \in \mathbb{S}^2$.
2. There is a point $z(k) \in \mathbb{S}^2$ such that $f(\gamma_k \cup \gamma_{k+1}) \subset \mathbb{B}(z(k), \eta)$.
3. Every simple closed curve $\gamma \subset [A(k, 1) - f^{-1}(\mathbb{B}(z(k), \eta))]$ bounds a disk in $A(k, 1)$.

Proof. In order to prove the first statement of the lemma, we distinguish four cases, depending on the position of $f(\gamma_k)$. We will use the fact that by Lemma 2.2, for $k \rightarrow \infty$, the curve $f(\gamma_k)$ becomes arbitrarily close to a point $z(k) \in \mathbb{S}^2$.

Case A: $f(\gamma_k) \subset D \in \mathcal{W}$.

In this case $f(A(k, 1))$ enters a component C of $\tilde{N} - \cup \mathcal{W}$ near $f(\gamma_k)$. Consider the compact component Z of $(f|_{A(k)})^{-1}(\overline{C}) \subset A(k)$ with boundary component γ_k and let α_k be the boundary curve of $Z - \gamma_k$ which is parallel to $\partial A(k) = \gamma_k$. By the definition of X_1 and X_2 , $\alpha_k = \gamma_{k+j} \in [X_1 \cup X_2] \subset X$, for some $j \geq 1$. By definition of X , $\gamma_{k+1} \subset A(k, j)$ and intersects the domain Z . If $\gamma_{k+1} \subset Z$, then clearly $f(\gamma_{k+1}) \subset \overline{C} \subset \chi_2(f(\gamma_k))$ and we are done. Otherwise, $f(\gamma_{k+1})$ must not be contained in $\tilde{N}(\Gamma)$. This means that γ_{k+1} belongs to X_3 and so it is contained in a face F of Λ , which intersects C . Hence, $F \subset \chi_2(f(\gamma_k))$ which implies $f(\gamma_{k+1}) \subset \chi_2(f(\gamma_k))$.

Case B: $f(\gamma_k) \subset [\partial \tilde{N}(\Gamma) - \cup \mathcal{W}]$ and the annulus $f(A(k, 1))$ enters $\tilde{N}(\Gamma)$ near $f(\gamma_k)$.

In this case, the arguments in Case A apply to show that $f(\gamma_{k+1}) \subset \chi_2(f(\gamma_k))$.

Case C: $f(\gamma_k) \subset [\partial \tilde{N}(\Gamma) - \cup \mathcal{W}]$ and the annulus $f(A(k, 1))$ enters $\mathbb{B} - \tilde{N}(\Gamma)$ near $f(\gamma_k)$.

First, note that if $f(\gamma_{k+1})$ intersects $\chi_2(f(\gamma_k))$, then $f(\gamma_{k+1}) \subset \chi_3(f(\gamma_k))$. Thus, we may assume that $f(\gamma_{k+1})$ lies outside the compact piecewise smooth ball $\chi_2(f(\gamma_k))$. Consider the compact component Z of $(f|_{A(k, 1)})^{-1}(\chi_2(f(\gamma_k)))$ containing γ_k in its boundary. Let $\alpha_k \neq \gamma_k$ be the boundary curve of Z which is parallel in $A(k)$ to γ_k ; recall that $A(k)$ is the end of A with boundary γ_k . If $f(\alpha_k)$ intersects $\cup \mathcal{L}$, then $f(\alpha_k)$ is contained in a disk $D \in \mathcal{L}$; in this case, since α_k lies between γ_k and γ_{k+1} and it is parallel to $\partial A(k)$, then $\alpha_k \in X_3$, which is contrary to the definition of γ_{k+1} . Thus, $f(\alpha_k) \subset \partial(\chi_2(f(\gamma_k)))$ and is disjoint from $\cup \mathcal{L}$, and so $f(\alpha_k) \subset \text{Int}(\tilde{N}(\Gamma))$. Let $A(\gamma_k, \alpha_k) \subset A(k, 1)$ be the subannulus with boundary curves $\gamma_k \cup \alpha_k$. As $f(A(\gamma_k, \alpha_k))$ enters

$\mathbb{B} - \tilde{N}(\Gamma)$ nears $f(\gamma_k)$ and $f(\alpha_k) \subset \text{Int}(\tilde{N}(\Gamma))$, then our previous separation arguments imply that there exists a curve $\beta \subset (f|_{A(\gamma_k, \alpha_k)})^{-1}(\partial\tilde{N}(\Gamma) - \cup\mathcal{W})$ which is parallel to γ_k . Since $\beta \in X_2$ and $\beta \neq \gamma_{k+1}$, we arrive at a contradiction. This contradiction proves Case C.

Case D: $f(\gamma_k) \subset D \in \mathcal{L}$.

If $f(\gamma_{k+1}) \subset \partial\tilde{N}(\Gamma)$ or $f(\gamma_{k+1}) \subset \hat{D} \in \mathcal{W}$, then the arguments in our previously considered cases imply that $f(\gamma_k) \subset \chi_3(f(\gamma_{k+1}))$. Hence, we may assume that $f(\gamma_{k+1}) \subset D' \in \mathcal{L}$ as well.

If $\chi_1(f(\gamma_k)) \cap \chi_1(f(\gamma_{k+1})) \neq \emptyset$, then $f(\gamma_{k+1}) \subset \chi_2(f(\gamma_k))$. Hence, we can assume that $\chi_1(f(\gamma_k)) \cap \chi_1(f(\gamma_{k+1})) = \emptyset$. Recall that $f|_{A(k,1)}$ is in general position with respect to $\partial(\chi_1(f(\gamma_k)))$ and $\partial(\chi_1(f(\gamma_{k+1})))$. Let Z_i be the component of $(f|_{A(k,1)})^{-1}(\chi_1(f(\gamma_i)))$ with boundary component γ_i and let $\alpha_i \neq \gamma_i$ be the boundary component of Z_i which is parallel to γ_i , for $i = k, k+1$, respectively. Since α_k and α_{k+1} lie in $\text{Int}(A(k,1))$, then by definition of X , both $f(\alpha_k)$ and $f(\alpha_{k+1})$ are disjoint from $\cup\mathcal{L}$. Moreover, as $f(\alpha_i) \subset \chi_1(f(\gamma_i))$, for $i = k, k+1$, then $f(\alpha_k \cup \alpha_{k+1}) \subset \text{Int}(\tilde{N}(\Gamma))$. Let $A(\alpha_k, \alpha_{k+1})$ be the subannulus of $A(k,1)$ with boundary $\alpha_k \cup \alpha_{k+1}$.

Consider the collection of disks $D_{\mathcal{W}}^k$ in \mathcal{W} which are contained in the interior of $\chi_2(f(\gamma_k)) - \chi_1(f(\gamma_k))$. Then $\tilde{N}(\Gamma) - \cup D_{\mathcal{W}}^k$ contains a connected domain whose closure Σ^k in \mathbb{B} satisfies $f(\alpha_k) \subset \text{Int}(\Sigma^k)$ and $f(\alpha_{k+1}) \subset \mathbb{B} - \Sigma^k$. Our previous separation arguments imply that there is a simple closed curve β in $(f|_{A(\alpha_k, \alpha_{k+1})})^{-1}(\partial\Sigma^k)$ which is parallel to γ_k . But $\beta \subset \text{Int}(A(k,1))$ and $\beta \in X_1 \cup X_2$, which is a contradiction. This contradiction completes the proof of the first statement of the lemma.

Item 1 in the lemma is a straightforward consequence of the fact that, as $\rightarrow \infty$, then $f(A(k))$ uniformly converges to \mathbb{S}^2 . Moreover, given a X_i -type curve $\gamma \subset A(k)$, $i = 1, 2, 3$, the Euclidean diameter of $f(\gamma)$ goes to zero (as $k \rightarrow \infty$) and is arbitrarily close to a point $y(\gamma)$ in \mathbb{S}^2 . Item 2 in the lemma follows from the observation that as $k \rightarrow \infty$, the sets $\chi_3(f(\gamma_k))$ are arbitrarily close to $f(\gamma_k)$, which in turn, lie arbitrarily close to points $z(k) \in \mathbb{S}^2$. These observations imply that there exists an integer $j(\eta)$ such that for $k \geq j(\eta)$, items 1 and 2 in Lemma 4.4 hold.

In order to obtain item 3, we define $k(\eta) = j(\frac{\eta}{900})$. By definition of $j(\frac{\eta}{900})$, for $k \geq k(\eta)$, $f(\gamma_k \cup \gamma_{k+1}) \subset \mathbb{B}(z(k), \frac{\eta}{900})$ and $f(A(k)) \subset [\mathbb{B} - \mathbb{B}(1 - \frac{\eta}{900})]$. It remains to check that each simple closed curve β in a component K of $(f|_{A(k,1)})^{-1}(\mathbb{B} - \mathbb{B}(z(k), \eta))$ bounds a disk in $A(k,1)$; note that $\overline{K} \subset \text{Int}(A(k,1))$. Observe that $\frac{\eta}{900}$ is sufficiently small so that there exist faces F_1, F_2, F_3 and F_4 in \mathcal{F} , such that: $\mathbb{B}(z(k), \frac{\eta}{900}) \subset \text{Int}(C(F_1))$, $C(F_i) \subset \text{Int}(F_{i+1})$, for $i = 1, 2, 3$ and $C(F_4) \subset \mathbb{B}(z(k), \eta)$.

If $\beta \subset K$ does not bound a disk in $A(k,1)$, then it is parallel to γ_k in $A(k,1)$. Let $A(\gamma_k, \beta)$ denote the subannulus of $A(k,1)$ with boundary $\gamma_k \cup \beta$. Then the arguments in the proof of Lemma 4.2 imply that there exists a simple closed curve $\gamma' \subset \text{Int}(A(\gamma_k, \beta))$ which is parallel to γ_k , $f(\gamma') \subset \mathbb{B}(z(k), \eta)$ and γ' is an X_i -type curve, for $i = 1, 2$ or 3 . In particular, $\gamma' \in X$ which is impossible. Thus, every simple closed curve in $A(k,1)$ whose image under f lies outside of $\mathbb{B}(z(k), \eta)$ bounds a disk in $A(k,1)$. This completes the proof of the lemma. \square

The next lemma directly follows from the mean curvature comparison principle.

Lemma 4.5 *Suppose $\Sigma \subset A$ is a compact domain such that: $f(\partial\Sigma)$ is contained in $\mathbb{B}(z, \eta)$, where $z \in \mathbb{S}^2$ and $\eta < \frac{1}{H_0}$. Then either $f(\Sigma) \subset \mathbb{B}(z, \eta)$ or $f(\Sigma)$ contains a point outside of $\mathbb{B}(z, \frac{1}{H_0})$.*

5 Proof of the Theorem 1.1.

By Lemma 3.4, $L(A) \subset \mathbb{S}^2$ contains at least two distinct points x and y . We next prove that the limit set of f is the entire sphere \mathbb{S}^2 .

Lemma 5.1 $L(A) = \mathbb{S}^2$.

Proof. By Lemma 3.4, there are distinct points $x, y \in L(A) \subset \mathbb{S}^2$. Arguing by contradiction, suppose that there exists a point $p \in \mathbb{S}^2 - L(A)$. The definition of limit point and the fact that $f: A \rightarrow \mathcal{D}$ is proper with $L(A) \subset \mathbb{S}^2$ imply there exists an $\varepsilon > 0$ such that $\mathbb{B}(p, 10\varepsilon) \cap f(A) = \emptyset$. By properness of f in \mathbb{B} , then for n large, we have $f(\overline{A - A_n}) \subset [\mathbb{B} - \mathbb{B}(1 - \varepsilon)]$.

Note that for some $\delta \in (0, \frac{1}{8}\varepsilon)$ sufficiently small, there exists a compact embedded annulus of revolution $E(\delta) \subset [(\overline{\mathbb{B}} - \mathbb{B}(1 - \delta)) \cap \mathbb{B}(p, \varepsilon)]$ with boundary circles in $\mathbb{S}^2 \cup [\mathbb{S}^2(1 - \delta)]$ and such that the radial projection $r(E(\delta)) \subset \mathbb{S}^2$ is contained in the disk $\mathbb{B}(p, \varepsilon) \cap \mathbb{S}^2$. Furthermore, $E(\delta)$ is also chosen to have mean curvature greater than H_0 and with mean curvature vector outward pointing from the domain in $[\overline{\mathbb{B}} - \mathbb{B}(1 - \delta)] - E(\delta)$ which is contained in $\mathbb{B}(p, \varepsilon)$; for instance, $E(\delta)$ can be chosen to be a piece of a suitably scaled compact embedded annulus in some nodoid of constant mean curvature one, see Figure 3 Left. Assume now that ε is also chosen less than $\frac{1}{10}d(x, y)$.

Assume that n and j are chosen sufficiently large so that:

1. $f(A(n, j)) \subset [\mathbb{B} - \overline{\mathbb{B}(1 - \delta)}]$.
2. Any circle in $\mathbb{S}^2 - \{x, y\}$ which represents the generator of the first homology group $\mathbb{H}_1(\mathbb{S}^2 - \{x, y\})$ and whose distance from x and y is at least δ , intersects the radial projection $r(f(A(n, j))) \subset \mathbb{S}^2$. This property holds since x and y are limit points of $f(A)$.
3. The radial projection of each of the two boundary curves of $f(A(n, j))$ has diameter less than ε . This condition is possible to achieve since each of the components of $\partial f(A(n, j))$ has image on either a disk component of \mathcal{W} , a face of \mathcal{F} or a component of $\partial \tilde{N}(\Gamma) - \cup \mathcal{W}$, and each of these components and faces is contained ambient balls of radius 4δ by Lemma 2.2, which in turn have radial projections of diameter less than ε .

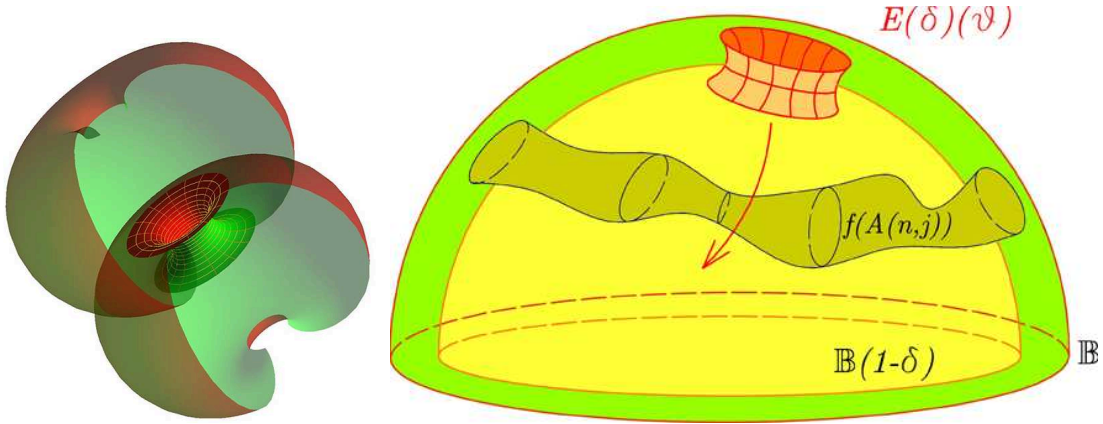


Figure 3: Left: This figure shows a domain on the nodoid corresponding to a scaling of $E(\delta)$. Right: Since all of the $E(\delta)(\vartheta)$ are disjoint from $\partial f(A(n, j))$, a first point of contact in $E(\delta)(\vartheta_0) \cap f(A(n, j))$ occurs an interior point of $f(A(n, j))$.

By the above three properties and our choices of ε and δ , we can choose a circle $S^1 \subset \mathbb{S}^2 - \{x, y\}$ which intersects $r(f(A(n, j)))$, and such that the ε -neighborhood $S^1(\varepsilon)$ of S^1 is disjoint from the

radial projection $r(\partial f(A(n, j))) \subset \mathbb{S}^2$ and each component of $\mathbb{S}^2 - r(S^1(\varepsilon))$ contains points of $r(f(A(n, j)))$. Let L be an oriented radial ray which is an axis for the circle S^1 . For $\vartheta \in [0, 2\pi)$, consider the family of annuli $E(\delta)(\vartheta)$ obtained by rotating $E(\delta)$ counterclockwise around L by the angle ϑ . By elementary separation properties, there is a smallest $\vartheta_0 \in (0, 2\pi)$ such that $E(\delta)(\vartheta_0) \cap f(A(n, j)) \neq \emptyset$. Since all of the $E(\delta)(\vartheta)$ are disjoint from $\partial f(A(n, j))$, a first point of contact in $E(\delta)(\vartheta_0) \cap f(A(n, j))$ occurs at an interior point of $f(A(n, j))$, which must have absolute mean curvature on A at least equal to the minimum of the mean curvature of $E(\delta)(\vartheta_0)$ (see Figure 3 Right). But the mean curvature of $E(\delta)(\vartheta_0)$ is greater than the absolute mean curvature function of A . This contradiction completes the proof of Lemma 5.1. \square

The next lemma follows immediately from the arguments presented in the proof of Lemma 5.1; also see Figure 3 Right. We note that the constant H_0 in the statement of the next lemma is the same constant which is the strict upper bound on the supremum of the absolute mean curvature of $f: A \rightarrow \mathbb{B}$.

Lemma 5.2 *Given any $\varepsilon \in (0, \frac{1}{4})$, there exists an $\eta_0 \in (0, \frac{\varepsilon}{10})$ that also depends on H_0 such that the following statements hold. For any $\eta \in (0, \eta_0]$ and for any immersion $g: \Sigma \rightarrow \mathbb{B} - \mathbb{B}(1 - \eta)$ of a compact surface with boundary and absolute mean curvature less than H_0 such that $g(\partial\Sigma) \subset [\overline{\mathbb{B}}(x, \eta) \cup \overline{\mathbb{B}}(y, \eta)]$ for two points $x, y \in \mathbb{S}^2$ with $d(x, y) \geq \varepsilon$, then either $g(\Sigma) \subset [\overline{\mathbb{B}}(x, \eta) \cup \overline{\mathbb{B}}(y, \eta)]$ or $g(\Sigma)$ is ε -close to every point in \mathbb{S}^2 . (Note that it may be the case that $g(\partial\Sigma)$ is contained entirely in one of the balls $\overline{\mathbb{B}}(x, \eta), \overline{\mathbb{B}}(y, \eta)$.)*

Lemma 5.3 *Given an $\varepsilon \in (0, \frac{1}{2H_0})$, there exists an $n(\varepsilon) \in \mathbb{N}$ such that for each $k \geq n(\varepsilon)$, there exists a point $y(k) \in \mathbb{S}^2$ with $f(A(k, 1)) \subset \mathbb{B}(y(k), \varepsilon)$ and $f(A(n(k))) \subset [\mathbb{B} - \overline{\mathbb{B}}(1 - \varepsilon)]$.*

Proof. Fix $\varepsilon \in (0, \frac{1}{2H_0})$ and let $\eta = \min\{\eta_0, \eta_1\}$ where η_0 is given in Lemma 5.2 and depends on ε and H_0 and η_1 given in Lemma 4.2. Let $k(\eta)$ be the related integer given in Lemma 4.4. We claim that for $k \geq k(\eta)$, $f(A(k, 1)) \subset \mathbb{B}(z(k), \eta)$ and that $f(A(k(\eta))) \subset [\mathbb{B} - \overline{\mathbb{B}}(1 - \varepsilon)]$, and so, by setting $n(\varepsilon) = k(\eta)$, this claim will complete the proof of the lemma. By Lemma 4.4, $f(A(k(\eta))) \subset [\mathbb{B} - \overline{\mathbb{B}}(1 - \eta)] \subset \mathbb{B} - \overline{\mathbb{B}}(1 - \varepsilon)$ and so it remains to verify that $f(A(k, 1)) \subset \mathbb{B}(z(k), \eta)$.

Suppose that $f(A(k, 1))$ contains a point outside of $\mathbb{B}(z(k), \eta)$. Let K be a nonempty component in $(f_{A(1,k)})^{-1}(\mathbb{B} - \mathbb{B}(z(k), \eta))$. By Lemma 4.5, there is a point on K which lies outside of $\mathbb{B}(z(k), \frac{1}{H_0})$. Since $\varepsilon \in (0, \frac{1}{2H_0})$ and $\eta \leq \eta_0$, Lemma 5.2 implies that the distance between every point of \mathbb{S}^2 and K is at most ε . In particular, there exists a point $p \in K$ such that $f(p)$ has distance greater than 1 from $z(k)$.

By the third statement in Lemma 4.4, each boundary curve of K bounds a disk in $A(k, 1)$. From the simple topology of an annulus we find that exactly one boundary curve of K bounds a disk $D \subset A(k, 1)$ and such that $K \subset D$. Next we apply Lemma 4.2 to find an X_i -type curve $\beta_1 \subset D$ which bounds a subdisk $D(\beta_1)$ which contains the point p and which satisfies the other properties in that lemma. In particular, we may assume the intrinsic distance from $D(\beta_1)$ to ∂D is at least η . By the second statement in Lemma 4.4, $f(\beta_1) \subset \mathbb{B}(y(\beta_1), \eta)$ for some point $y(\beta_1) \in \mathbb{S}^2$. By our previous arguments there exists a point $p_1 \in D(\beta_1)$ such that the distance from $f(p_1)$ to $y(\beta_1)$ is greater than one. So, we can apply Lemma 4.2 again to obtain a subdisk $D(\beta_2)$, $D \supset D(\beta_1) \supset D(\beta_2)$, where the intrinsic distance from $\partial D(\beta_1)$ to $D(\beta_2)$ is at least η . Repeating these arguments, induction gives the existence of a sequence of disks $D \supset D(\beta_1) \supset \dots \supset D(\beta_n) \supset \dots$ such that the intrinsic distance from $D(\beta_n)$ to ∂D is at least $n\eta$. Since D is compact, we obtain a contradiction which proves our

earlier claim that $f(A(k, 1)) \subset \mathbb{B}(z(k), \eta)$ for $k \geq k(\eta)$. As we have already observed, this claim then proves the lemma. \square

We now complete the proof of Theorem 1.1. Fix some $\varepsilon' \in (0, \frac{1}{2H_0})$ and let $\eta_0 \in (0, \frac{\varepsilon'}{10})$ be the related number given in Lemma 5.2. Set $\varepsilon = \eta_0$ and let $n(\varepsilon)$ be the integer given in Lemma 5.3. By Lemma 5.3, for each $i \in \mathbb{N}$, $f(A(n(\varepsilon), i)) \subset [\mathbb{B} - \mathbb{B}(1 - \eta_0)]$, $f(\gamma_{n(\varepsilon)}) \subset \mathbb{B}(y(n(\varepsilon)), \eta_0)$ and $f(\gamma_{n(\varepsilon)+i}) \subset \mathbb{B}(y(n(\varepsilon) + i), \eta_0)$. Since the limit set of $A(n(\varepsilon))$ is all of \mathbb{S}^2 , there exists a smallest $j \in \mathbb{N}$ such that the distance between $y(n(\varepsilon))$ and $y(n(\varepsilon) + j)$ is greater than ε' . By Lemma 5.2 and taking into account that $\mathbb{B}(y(n(\varepsilon)), \eta_0)$ and $\mathbb{B}(y(n(\varepsilon) + j), \eta_0)$ are disjoint, then we conclude that $f(A(n(\varepsilon), j))$ must be ε' close to every point of \mathbb{S}^2 .

On the other hand, given $k \in \mathbb{N}$, $n(\varepsilon) \leq k < n(\varepsilon) + j$, we know (by Lemma 5.3) that $f(A(k, 1)) \subset \mathbb{B}(y(k), \varepsilon)$, for a suitable $y(k) \in \mathbb{S}^2$. Moreover, the choice of j implies that $f(\gamma_k) \subset \mathbb{B}(y(n(\varepsilon)), \varepsilon' + \varepsilon)$, for k satisfying $n(\varepsilon) \leq k < n(\varepsilon) + j$. So, by the triangle inequality we deduce $f(A(k, 1)) \subset \mathbb{B}(y(n(\varepsilon)), \varepsilon' + 2\varepsilon) \subset \mathbb{B}(y(n(\varepsilon)), 2\varepsilon')$ for any k satisfying $n(\varepsilon) \leq k < n(\varepsilon) + j$. This implies $f(A(n(\varepsilon), j)) \subset \mathbb{B}(y(n(\varepsilon)), 2\varepsilon')$ which is impossible since $2\varepsilon' < \frac{1}{10}$ and we have already seen that $f(A(n(\varepsilon), j))$ must be ε' close to every point of \mathbb{S}^2 . This contradiction completes the proof of Theorem 1.1 in the case $\overline{W} = \overline{\mathbb{B}}$.

For the general case where \overline{W} is a smooth compact Riemannian manifold with nonempty boundary, small modifications of the proof of Theorem 1.1 in the special case $\overline{W} = \overline{\mathbb{B}} \subset \mathbb{R}^3$ also demonstrate that there exists a properly embedded 1-manifold $\Delta_W \subset W$, whose path components are smooth simple closed curves, such that $\mathcal{D} = W - \Delta_W$ is a Calabi-Yau domain for any open surface with at least one annular end. In carrying out these modifications in the smooth compact 3-manifold \overline{W} , it is convenient, to place the 1-manifold $\Delta_{\overline{W}}$ in the union of small pairwise disjoint closed ε -neighborhoods of the boundary components of \overline{W} which have a natural product structure derived from the distance function to the boundary component. The product structure simplifies the construction of the related 1-complex $\Gamma_{\overline{W}}$ which has one component in each of the ε -neighborhoods of each boundary component of \overline{W} . Also note that the properness of any proper immersion of $A = \mathbb{S}^1 \times [0, \infty)$ into \mathcal{D} guarantees that A has a end representative which maps into the ε -neighborhood of exactly one of the boundary components of \overline{W} . This discussion completes the proof of Theorem 1.1.

Remark 5.4 The reader familiar with the paper [6] might consider the question: Are the domains $\mathcal{D}_{\mathcal{F}} \subset \mathbb{R}^3$ [6], obtained by removing a infinite proper family \mathcal{F} of horizontal circles from \mathbb{B} , Calabi-Yau domains for surfaces with at least one annular end? The answer to this question is no because for at least one such $\mathcal{D}_{\mathcal{F}}$ constructed in [6], there exists a proper, conformal, complete embedding $f: \mathbb{R}^2 \rightarrow \mathcal{D}$ with absolute mean curvature function less than 1, $f(\mathbb{R}^2)$ is a surface of revolution with axis the x_3 -axis, $f(\mathbb{R}^2)$ has intrinsic linear area growth and has limit set $L(\mathbb{R}^2) = \mathbb{S}^2$. The mean curvature function of $f(\mathbb{R}^2)$ in this case contains points of mean curvature arbitrarily close to 1 and also arbitrarily close to -1 . In this case for \mathcal{D} , the circles in $\mathcal{F} \subset \mathbb{B}$ are chosen to have axis the x_3 -axis; the surface has the appearance of taking an infinite connected sum of the spheres \mathbb{S}_k^2 , $k \in \mathbb{N}$, defined at the beginning of Section 2, joined by small catenoidal type necks centered along points along the x_3 -axis which limit to the north and south poles of \mathbb{S}^2 .

We conclude the paper with the following conjecture.

Conjecture 5.5 *Let $\Delta \subset \mathbb{B}$ be the properly embedded one-manifold given in the proof of Theorem 1.1. If \overline{B} is a smooth compact Riemannian three-ball and $F: \overline{B} \rightarrow \mathbb{B}$ is a smooth diffeomorphism, then $\mathcal{D} = B - F^{-1}(\Delta)$ is a Calabi-Yau domain for **any** noncompact surface with compact boundary*

(possibly empty). In particular, $\mathcal{D} = \mathbb{B} - \Delta$ does not admit any complete, properly immersed open surfaces with bounded mean curvature.

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